

On the Generalized Arcsine Probability Distribution with Bounded Support

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Abstract

Properties of the beta functions are investigated. We define the generalized arcsine probability distribution with bounded support. The properties of the beta functions prove some results for this distribution.

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1 Introduction

The arcsine distribution plays main role in many fields. For example, the arcsine laws are a collection of results for one-dimensional random walks and Brownian motion (the Wiener process, see [1]). All three laws relate path properties of the Wiener process to the arcsine distribution. In particular, the third arcsine law states that the time at which a Wiener process achieves its maximum is arcsine distributed.

Definition 1.1. A random variable X has the standard arcsine distribution if X has probability density function f given by

$$f(x) = \frac{1}{\sqrt{x(1-x)}}, x \in (0, 1).$$

Definition 1.2. The beta function is defined as

$$\beta(s, t) = \int_0^1 u^{s-1}(1-u)^{t-1} du,$$

for $\Re(s) > 0$ and $\Re(t) > 0$.

Theorem 1.3. *The central moments of a random variable X with the standard arcsine distribution are*

$$\mu_k = \begin{cases} 0, & k=2n-1; \\ \frac{(2n)!}{16^n(n!)^2} & k=2n. \end{cases}$$

where $n \in \mathbb{N}$.

This Theorem is a special case of Theorem 2.5.

2 Additional properties of beta functions

Theorem 2.1. *For $\Re(s) > 0$, $\Re(t) > 0$, and any two real numbers $r_1 < r_2$. The beta function is give as*

$$\beta(s, t) = \frac{1}{(r_2 - r_1)^{s+t-1}} \int_{r_1}^{r_2} (x - r_1)^{s-1} (r_2 - x)^{t-1} dx$$

Proof. Using the substitution $w = \frac{x-r_1}{r_2-r_1}$ to get that

$$\begin{aligned} \int_{r_1}^{r_2} (x - r_1)^{s-1} (r_2 - x)^{t-1} dx &= \int_0^1 ((r_2 - r_1)w)^{s-1} (r_2 - r_1 - (r_2 - r_1)w)^{t-1} (r_2 - r_1) dw \\ &= (r_2 - r_1)^{s+t-1} \int_0^1 w^{s-1} (1-w)^{t-1} dw \\ &= (r_2 - r_1)^{s+t-1} \beta(s, t). \end{aligned}$$

□

Moreover, the beta functions satisfy

Theorem 2.2. For $r_1 < \frac{r_1+r_2}{2} = \bar{r} < r_2$, $\Re(s) > 0$, and $\Re(t) > 0$

$$\int_{\bar{r}}^{r_2} (x - r_1)^{s-1} (x - \bar{r})^{t-1} (r_2 - x)^{s-1} dx = \frac{1}{2} \left(\frac{r_2 - r_1}{2} \right)^{2s+t-2} \beta(s, \frac{t}{2}).$$

Moreover,

$$\int_{r_1}^{\bar{r}} (x - r_1)^{s-1} (\bar{r} - x)^{t-1} (r_2 - x)^{s-1} dx = \frac{1}{2} \left(\frac{r_2 - r_1}{2} \right)^{2s+t-2} \beta(s, \frac{t}{2}).$$

In particular, the beta function can be defined as

$$\beta(s, t) = \int_0^{1/2} x^{s-1} \left(\frac{1}{2} - x \right)^{2t-1} (1-x)^{s-1} = \int_{1/2}^1 x^{s-1} \left(x - \frac{1}{2} \right)^{2t-1} (1-x)^{s-1}.$$

Proof. Using the substitution $w = \left(\frac{2(x-\bar{r})}{r_2-r_1} \right)^2$ and the fact that

$$\bar{r} - r_1 = r_2 - \bar{r} = \frac{r_2 - r_1}{2}$$

to get that

$$\begin{aligned} & \int_{\bar{r}}^{r_2} (x - r_1)^{s-1} (x - \bar{r})^{t-1} (r_2 - x)^{s-1} dx \\ &= \int_0^1 \left(\bar{r} - r_1 + \frac{r_2 - r_1}{2} \sqrt{w} \right)^{s-1} \left(\frac{r_2 - r_1}{2} \sqrt{w} \right)^{t-1} \left(r_2 - \bar{r} - \frac{r_2 - r_1}{2} \sqrt{w} \right)^{s-1} dw \\ &= \frac{1}{2} \left(\frac{r_2 - r_1}{2} \right)^{2s+t-2} \int_0^1 (1-w)^{s-1} w^{\frac{t}{2}-1} dw = \frac{1}{2} \left(\frac{r_2 - r_1}{2} \right)^{2s+t-2} \beta(s, \frac{t}{2}). \end{aligned}$$

Now, following the same proof with $\sqrt{w} = \frac{2(\bar{r}-x)}{r_2-r_1}$ to prove the assertion

$$\int_{r_1}^{\bar{r}} (x - r_1)^{s-1} (\bar{r} - x)^{t-1} (r_2 - x)^{s-1} dx = \frac{1}{2} \left(\frac{r_2 - r_1}{2} \right)^{2s+t-2} \beta(s, \frac{t}{2}).$$

□

Example 2.3. using $r_1 = 1$, $r_2 = 3$, $s = t = 3/2$ and $\bar{r} = 2$ to get that

$$\begin{aligned} \int_2^3 \sqrt{6 + 6x^2 - 11x - x^3} dx &= \int_2^3 \sqrt{(3-x)(x-2)(x-1)} dx \\ &= \frac{1}{2} \beta(3/2, 3/4) \cong 0.47925609389. \end{aligned}$$

Also, using $r_1 = -1$, $r_2 = 3$, $t = 7$, $s = 1/2$ and $\bar{r} = 1$ to get that

$$\int_{-1}^1 \frac{(1-x)^6}{\sqrt{3+2x-x^2}} dx = \int_{-1}^1 (1-x)^6 (3-x)^{-1/2} (1+x)^{-1/2} dx = \frac{1}{2} (2)^6 \beta(1/2, 7/2) = 10\pi$$

2.1 The generalized arcsine probability distribution with bounded support

Using Theorem 2.1, we can easily show that for $s, t > 0$, the function

$$f(x) = \frac{(x-r_1)^{s-1} (r_2-x)^{t-1}}{(r_2-r_1)^{s+t-1} \beta(s, t)}, r_1 < x < r_2,$$

and zero elsewhere is probability density function. Moreover, this Theorem implies that the mean of the random variable X is

$$\begin{aligned} \mu = E(X) &= \int_{r_1}^{r_2} x \frac{(x-r_1)^{s-1} (r_2-x)^{t-1}}{(r_2-r_1)^{s+t-1} \beta(s, t)} dx \\ &= \int_{r_1}^{r_2} \frac{(x-r_1)^s (r_2-x)^{t-1}}{(r_2-r_1)^{s+t-1} \beta(s, t)} dx \\ &\quad + r_1 \int_{r_1}^{r_2} \frac{(x-r_1)^{s-1} (r_2-x)^{t-1}}{(r_2-r_1)^{s+t-1} \beta(s, t)} dx = r_1 + \frac{s}{s+t} (r_2-r_1). \end{aligned}$$

Consider the following special case when $s = t$

Definition 2.4. The random variable X has is a generalized arcsine random variable if the probability density function is support

$$f(x) = \frac{((x-r_1)(r_2-x))^{s-1}}{(r_2-r_1)^{2s-1} \beta(s, s)}, r_1 < x < r_2.$$

The above argument shows that this function is a valid density function. Moreover, it implies that in this case, when $s = t$, $\mu = \frac{r_2+r_1}{2} = \bar{r}$.

As a special case, if $s = 1/2$ then this distribution is called the arcsine probability distribution with bounded support (see [2]). $s = 1/2$ and $r_1 = 0$, $r_2 = 1$. Then X has the standard arcsine distribution.

Theorem 2.5. *The central moments of a random variable X with the standard arcsine distribution are*

$$\mu_k = \frac{((-1)^k + 1)(r_2 - r_1)^k \beta(s, \frac{k+1}{2})}{2^{2s+k} \beta(s, s)} = \begin{cases} 0, & k=2n-1; \\ \frac{(r_2-r_1)^{2n} \beta(s, n+\frac{1}{2})}{2^{2s+2n-1} \beta(s, s)} & k=2n. \end{cases}$$

where $n \in \mathbf{N}$.

Proof.

$$\begin{aligned} \mu_k &= E(X - \mu)^k = \int_{r_1}^{r_2} (x - \mu)^k \frac{((x - r_1)(r_2 - x))^{s-1}}{(r_2 - r_1)^{2s-1} \beta(s, s)} dx \\ &= \int_{r_1}^{r_2} (x - \bar{r})^k \frac{((x - r_1)(r_2 - x))^{s-1}}{(r_2 - r_1)^{2s-1} \beta(s, s)} dx \\ &= \frac{1}{(r_2 - r_1)^{2s-1} \beta(s, s)} \left(\int_{r_1}^{\bar{r}} (x - \bar{r})^k ((x - r_1)(r_2 - x))^{s-1} dx \right. \\ &\quad \left. + \int_{\bar{r}}^{r_2} (x - \bar{r})^k ((x - r_1)(r_2 - x))^{s-1} dx \right). \end{aligned}$$

Using Theorem 2.2,

$$\begin{aligned} \mu_k &= \frac{1}{(r_2 - r_1)^{2s-1} \beta(s, s)} ((-1)^k \int_{r_1}^{\bar{r}} (\bar{r} - x)^k ((x - r_1)(r_2 - x))^{s-1} dx \\ &\quad + \int_{\bar{r}}^{r_2} (x - \bar{r})^k ((x - r_1)(r_2 - x))^{s-1} dx) \\ &= \frac{((-1)^k + 1)(r_2 - r_1)^k \beta(s, \frac{k+1}{2})}{2^{2s+k} \beta(s, s)}. \end{aligned}$$

In other words, for $n \in \mathbf{N}$, we have $\mu_{2n-1} = 0$ and $\mu_{2n} = \frac{(r_2-r_1)^{2n} \beta(s, n+\frac{1}{2})}{2^{2s+2n-1} \beta(s, s)}$.

Consequently, If X has arcsine probability distribution with bounded support then $\mu_{2n-1} = 0$ and using the fact that $\Gamma(n + \frac{1}{2}) = \frac{(2n)! \sqrt{\pi}}{4^n n!}$

$$\mu_{2n} = \frac{(r_2 - r_1)^{2n} \beta(\frac{1}{2}, n + \frac{1}{2})}{4^n \beta(\frac{1}{2}, \frac{1}{2})} = \frac{(r_2 - r_1)^{2n} (2n)!}{16^n (n!)^2}.$$

□

For the standard arcsine distribution, the central moments are $\mu_{2n-1} = 0$ and $\mu_{2n} = \frac{(2n)!}{16^n (n!)^2}$.

References

- [1] Morters, P. and Peres, Y. (2010), *Brownian motion*, Vol. 30. Cambridge Series in Statistical and Probabilistic Mathematics.
- [2] Rogozin, B.A. (2001), in Hazewinkel, Michiel, *Encyclopedia of Mathematics*, Springer, ISBN 978-1-55608-010-4.